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NOTES ON MAGNETO-HYDRODYNAMICS - NUMBER IV

OHM'S LAW

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Preface

In this note (an extension of the second note in the original series) the question of the formal nature and domain of validity of Ohm's law is considered. The problem is difficult, and the results cannot be claimed to be definitive. One of the complicating factors turns out to be the presence of two entirely distinct mechanisms, namely collisions and gyro oscillations, which tend to validate Ohm's law under entirely distinct conditions. The analysis is macroscopic, but is based on equations of momentum conservation which are consistent with microscopic analysis. An important formal point is the clarification of the concept of different electrical conductivities parallel and perpendicular to the magnetic field. It is shown that there are two possible definitions for "conductivity", one of which has the conventional property that its transverse component varies as $1/B^2$ for large B, while the other (more natural) definition implies a conductivity which is essentially independent of B. Of course, proper use of either definition, in combination with the Hall effect must yield the same results.

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Ohm's Law

1. Qualitative Considerations

Maxwell's equations contain the vectors E, B, D, H, J and the scalar q. In a purely electromagnetic problem with $B = \mu H$, $D = \kappa E$, and $q = J = 0$, these equations by themselves are a determined system associated with well-posed initial and boundary value problems. The same statement holds if q and J are given as functions of x and t subject only to $\partial q/\partial t + \operatorname{div} J = 0$. However, if we seek a formulation in which J also is an unknown, it is necessary to supplement Maxwell's equations by an additional vector equation; (the further determination of q requires only initial data, using charge conservation). This additional equation is frequently taken to be Ohm's law,

$$J = \sigma E \tag{1}$$

where σ, the electrical conductivity, is a known scalar constant (or function of x or of the local electromagnetic field).

A similar counting of variables and equations shows the necessity of a supplementary relation such as Ohm's law in the simplest magneto-hydrodynamic formulation. Consider a single-fluid formulation with mass, momentum, and energy conservation equations in addition to Maxwell's equations. The fluid equations (without stresses or heat flow) are a determined system when supplemented by an equation of state provided that we ignore the electromagnetic terms. Including the electromagnetic terms and Maxwell's equations reduces the degree of determinacy to that of the purely electromagnetic case, insofar as we can rely on the mere counting of variables.

Next, let us consider a two-fluid theory. The simplest such theory involves two mass-conservation equations, while retaining a single momentum and a single energy equation for the total fluid. We assume that a single equation of state involving energy, pressure, and both densities is given. Counting variables, compared to the one-fluid theory, we have an extra density (matched by an extra mass-conservation equation) and also an extra velocity vector, e.g., the difference or diffusion velocity. In ordinary fluid dynamics, this extra velocity is taken care of by supplementing the system of conservation equations by a diffusion relation, e.g., an assertion that the difference velocity is proportional to the concentration gradient. If we adopt this procedure and now adjoin the electromagnetic field, we discover that the counting is complete and there is no necessity for Ohm's law in virtue of the identities

$$\begin{cases} J = q_1 u_1 + q_2 u_2 \\ q = q_1 + q_2 \\ q_r = \gamma_r \rho_r, \quad \gamma_r = \varepsilon_r/m_r, \quad r = 1, 2. \end{cases}$$

Here, we can consider either a diffusion velocity or the electric current as an extra variable, but not both. In other words, the single vector relation with which we must supplement the equations of conservation and Maxwell's equations may be expected to have a connection with both diffusion and Ohm's law.

Finally, let us consider a more elaborate two-fluid theory with two distinct momentum equations as well as two mass equations, still keeping a single energy equation. In comparison to the preceding case, we have a single additional variable, namely a pressure (there are now two) and it is conventional to supplement this system of equations by an extra equation of state. We are left with an extra vector equation (difference momentum) so no diffusion relation or relation similar to Ohm's law can be adjoined. We shall see that the diffusion relation and Ohm's law are both contained in this difference momentum equation as certain special limiting cases, and, if the complexity of the situation warrants it, the difference momentum equation can be taken instead of Ohm's law.

It is illuminating to examine this difference momentum equation qualitatively. The crucial feature of a conducting fluid is that it contains two types of particles (or fluids) with distinct, in fact, opposite electrical properties, namely positive and negative charges. An electric field produces an instantaneous and local acceleration of the two types of particle (or fluid) in opposite directions. If we imagine a friction mechanism dependent primarily on the relative motion of the two fluids (this is consistent with the particle model and inter-particle collisions), a quasi-equilibrium is reached in which the relative velocity is, say, proportional to the applied electric field. This, however, is Ohm's law, since J is given by the relative velocity of the two fluids. This is also the conventional diffusion mechanism, except that, in the absence of electrical forces, the mechanism producing relative velocity usually depends on the difference in mass of the two types of particles and is likely to be a much smaller effect.

This oversimplified picture will obviously be modified by consideration of the inertia of the particles (i.e., the quasi-equilibrium is not set up instantaneously), by the presence of a magnetic field (the particle motion is not simply an acceleration in the direction of E), by the interaction with other diffusion producing mechanisms such as pressure or concentration gradients, other fields such as gravity, by a more complex frictional law than proportionality to the mean relative velocity, by phenomena such as ionization and recombination. It is clear that all of these complications (except the last) are contained in an appropriately chosen relative momentum equation, since this expresses exactly the relevant phenomenon, differential acceleration of the two fluids.

We can now compare more precisely the one- and two-fluid theories. Any one-fluid theory which incorporates the variable J should, perhaps, be called a "1 1/2-fluid" theory, since $J \neq 0$ implies a relative velocity, and $q \neq 0$ implies unequal number density of two fluids, even if they are not explicitly referred to. However, in the previous note, MH-III, it was noted that, because of the numerically large value of ε/m, it frequently happens that an appreciable charge q can be observed while the ratio of positive to negative number density, ν_+/ν_-, is very close to one; (specifically, when the Debye length is small compared to relevant dimensions). Furthermore, under somewhat more restrictive, but still fairly general conditions, a macroscopically appreciable current can be observed even though the relative velocity between positive and negative fluids is quite small. This is the justification for a "1 1/2-fluid" theory in which q and J are kept in Maxwell's equations and yet a single velocity and density are used to describe the fluid. In general, we might expect that a one-fluid theory coupled with a quasi-equilibrium relation such as Ohm's law would be valid when the diffusion velocities are small, while a full two-fluid formulation is required when the diffusion velocities are not small;

this is a rough rule-of-thumb which we will find needs modification in detail.

We shall use the term "Ohm's law" rather loosely to refer to any quasi-equilibrium relation which bears some similarity to (1) but shall drop this appelation for the more complex forms which include inertia, the principle reason being a change in mathematical structure, and, particularly in the initial data required in the two cases.

In the qualitative argument given above connecting Ohm's law with diffusion as caused by E, it would seem appropriate to state the result in a coordinate system moving with the local fluid velocity; this would suggest the form

$$(2) \qquad J_u \equiv J - qu = \sigma(E + u \times B) \equiv \sigma E_u$$

for Ohm's law. This form is consistent with the interpretation of resistivity as a dissipative mechanism (the entropy increases, cf. equation MH-I(22)*), and will, indeed, turn out to be the simplest of the generalizations of (1) appropriate to a moving fluid.

In the above discussion, the basic mechanism producing a quasi-equilibrium is friction, that is particle collisions. It will be seen, however, (in this note by a macroscopic analysis of the difference momentum equation --

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*Refers to Notes on Magneto-Hydrodynamics - Number I, equation number 22.

later by analysis of particle trajectories) that, under certain conditions, gyro oscillations provide an entirely independent mechanism leading to a form of Ohm's law. In other words, something like Ohm's law is valid in two entirely different domains of the physical parameters, and collisions are crucial for only one of these.

2. Formal Analysis

Our procedure will be to start from the momentum equations, and, in a purely formal way at first, to show how various forms of Ohm's law are obtained by dropping selected terms from these equations. Afterwards, we shall return to the full equations and endeavor to describe physical situations under which the dropped terms are indeed small. By its very nature, this is not a strictly justifiable mathematical procedure, since the momentum equations are only a part of a very complex coupled system of equations. Also, we shall occasionally find it necessary to make use beforehand of results and estimates which will be treated in detail only in later notes.

We shall assume that the frictional terms are proportional to velocity differences. This is intuitively valid as a rough approximation, and can be quantitatively justified by consideration of the Boltzmann equation,* as a good approximation under fairly general circumstances. Whenever necessary, the stress tensor will be replaced by a scalar pressure, although many of the arguments are insensitive to this approximation; (this matter will be gone into in great detail subsequently). Also, we assume that each fluid is charged, i.e., there are no neutral particles. The modification to include an additional neutral fluid is straightforward, but the quantitative results would be of rather doubtful utility without considering ionization and recombination processes.

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* See, Kolodner, I. I., On the Application of Boltzmann Equations to the Theory of Gas Mixtures. NYU Doctoral Dissertation, (1950).

The momentum equations are

$$(3) \qquad \frac{\partial u_r^i}{\partial t} + u_r^j \frac{\partial u_r^i}{\partial x^j} + \frac{1}{\rho_r} \frac{\partial P_r^{ij}}{\partial x^j} = \frac{X_r^i}{\rho_r}$$

and the difference momentum equations are

$$(4) \quad \frac{\partial v_r^i}{\partial t} + u_r^j \frac{\partial u_r^i}{\partial x^j} - u^j \frac{\partial u^i}{\partial x^j} + \frac{1}{\rho_r} \frac{\partial P_r^{ij}}{\partial x^j} - \frac{1}{\rho} \frac{\partial P^{ij}}{\partial x^j} = \frac{X_r^i}{\rho_r} - \frac{X^i}{\rho}$$

(cf. equations MH-I(3),(5)). We take

$$(5) \qquad \begin{aligned} X_r &= q_r \, (E + u_r \times B) - \sum_s \alpha_{rs} \, (u_r - u_s) \\ &= q_r \, (E_u + v_r \times B) - \sum_s \alpha_{rs} \, (v_r - v_s) \end{aligned}$$

where

$$(6) \qquad E_u = E + u \times B \quad .$$

Equations (4) are not independent; taking each one with weight ρ_r, their sum vanishes identically. This fact is relevant if we consider (4), in some sense, as a set of equations for v_r since the v_r also are not independent and satisfy the identity $\sum \rho_r v_r = 0$. Note that

$$(7) \qquad X = \sum X_r = q\,E + J \times B$$
$$= q\,E_u + J_u \times B$$

where

$$(8) \qquad J_u = J - q\,u \;.$$

In the two-fluid case, J_u and u can be introduced as variables instead of u_1 and u_2, and we obtain (cf. equations MH-I(13),(13a),

$$(9) \quad \frac{\partial J_u^i}{\partial t} + \frac{\partial}{\partial x^j}(J_u^i u^j) + J_u^j \frac{\partial u^i}{\partial x^j} + \frac{\partial}{\partial x^j} \left(\frac{\gamma_1 \rho_2 + \gamma_2 \rho_1}{(\gamma_1 - \gamma_2)^2 \rho_1 \rho_2} J_u^i J_u^j \right) + \sum_r \gamma_r$$
$$\frac{\partial P_r^{ij}}{\partial x^j} - \frac{q}{\rho} \frac{\partial P^{ij}}{\partial x^j} = (\gamma_1 - \gamma_2)^2 \frac{\rho_1 \rho_2}{\rho} E_u - \frac{\alpha \rho}{\rho_1 \rho_2} J_u + (\rho_1 \gamma_2 + \rho_2 \gamma_1) J_u \times B$$

instead of (4); $\alpha = \alpha_{12} = \alpha_{21}$.

To start, we assume that all inertial and pressure terms can be ignored compared to electromagnetic and frictional terms, in which case (4) reduces to

$$X_r / \rho_r = X / \rho$$

or, from (5) and (7)

$$(10) \quad (\rho q_r - \rho_r q)\, E_u + (\rho q_r v_r - \rho_r J_u) \times B$$
$$- \rho \sum_s \alpha_{rs} (v_r - v_s) = 0, \quad r = 1,\, 2,\, \ldots \quad .$$

If we consider ρ_r, q_r, E_u, B, α_{rs} as given, this is a set of linear equations for the velocities v_r (also contained implicitly in $J_u = \sum q_r v_r$). As mentioned previously, the counting is correct since the v_r satisfy a linear homogeneous relation and the equations (10) are not independent.

The solution expresses v_r as a homogeneous linear function of E_u with coefficients depending on ρ_r, q_r, α_{rs}, and B. The same is therefore true of $J_u = \sum q_r v_r$. The most general linear homogeneous vector function of a vector E_u with coefficients dependent on scalars and an axial vector B can be written

$$(11) \qquad J_u = aE_u + bE_u \times B + c\,(E_u \times B) \times B$$

and this can be inverted in the form,

$$(12) \qquad E_u = R_{\parallel} J_u + \xi J_u \times B + (R_{\parallel} - R_{\perp}) \frac{(J_u \times B) \times B}{B^2} \quad .$$

$R_{\parallel}$, $R_{\perp}$, and ξ can be written explicitly in terms of a, b, and c, as follows:

$$R_{\parallel} = \frac{1}{a} \ , \quad \xi = \frac{-\,b}{(a-cB^2)^2+b^2B^2} \ , \quad R_{\perp} = \frac{a-cB^2}{(a-cB^2)^2+b^2B^2} \ ,$$

and

$$a = \frac{1}{R_{\parallel}} \ , \quad b = \frac{-\,\xi}{R_{\perp}^2+\xi^2B^2} \ , \quad c = \frac{1}{B^2}\left\{ \frac{1}{R_{\parallel}} - \frac{R_{\perp}}{R_{\perp}^2+\xi^2B^2} \right\}$$

All of these are scalar functions of ρ_r, q_r, α_{rs}, and $|B|$ (the magnitude of the magnetic field).

The reason for the notation adopted in (12) is as follows. If E_u is parallel to B, the same must be true of J_u, and we have

(13) $$E_u = R_{\parallel} J_u$$

If E_u is perpendicular to B, then the same must be true of J_u, and we have

(14) $$E_u = R_{\perp} J_u + \xi J_u \times B \quad ;$$

ξB is sometimes called the Hall vector.

It is worthwhile to study the dependence of $R_{\parallel}$, $R_{\perp}$, and ξ on α_{rs} and $|B|$. From (10), we can conclude that each of $R_{\parallel}$, $R_{\perp}$, and $\xi|B|$ is a homogeneous function of first degree in B and α_{rs}, i.e., if we consider $\alpha_{rs} = \alpha\sigma_{rs}$ with σ_{rs} fixed, each takes either of the forms $B\varphi(\alpha/B)$ or $\alpha\psi(B/\alpha)$. Actually, $R_{\parallel}$ is independent of B and can be obtained by inversion of the matrix

$$\beta_{rs} = \alpha_{rs} - \delta_{rs} \sum_s \alpha_{rs} \quad .$$

$R_{\perp}$ and ξ vary between finite limits as $0 < B < \infty$. At $B = 0$, we have $R_{\perp} = R_{\parallel}$ and $\xi|B| = 0$, while for $B \to \infty$ we get

(15) $$\begin{cases} \dfrac{1}{\xi} = \dfrac{\rho^2}{\sum \rho_r/\gamma_r} - q \\[2ex] R_{\perp} = \dfrac{\rho^2 \sum\limits_{r,s} \alpha_{rs}\left(\dfrac{1}{\gamma_r} - \dfrac{1}{\gamma_s}\right)^2}{2\,\xi^2\,(\sum \rho_r/\gamma_r)^2} \end{cases} \quad ;$$

this computation is given in the appendix.

Finally, we remark that, for the case of only two fluids, we obtain particularly simple and explicit results. In particular, $R_{\perp} \equiv R_{\parallel} = R$ so (12) becomes

(16) $$E_u = R J_u + \xi J_u \times B$$

and ξ as well as R is independent of B, explicitly

$$(17) \quad \begin{cases} \xi = -(\gamma_1/\rho_1 + \gamma_2/\rho_2)/(\gamma_1 - \gamma_2)^2 \\ R = \alpha\,(1/\rho_1 + 1/\rho_2)^2/(\gamma_1 - \gamma_2)^2 \end{cases}$$

These results can also be obtained by inspection from equation (9).

The inversion of (16) can be put in the form

$$(18) \qquad J_u = \frac{1}{R} E_u^{\parallel} + \frac{R}{R^2+\xi^2B^2} E_u^{\perp} - \frac{\xi}{R^2+\xi^2B^2} E_u^{\perp} \times B$$

where

$$E_u = E_u^{\parallel} + E_u^{\perp}$$

$$E_u^{\parallel} = \frac{B(B\cdot E_u)}{B^2} \quad , \quad E_u^{\perp} = \frac{B\times(E_u\times B)}{B^2}$$

From (16) we would be tempted to call R the resistivity and $\xi J_u \times B$ the "Hall effect." From (18) (which is mathematically identical to (16)) we would be tempted to call $1/R$ the conductivity in the direction B, $R/(R^2+\xi^2B^2)$ the conductivity perpendicular to B, and the last term the "Hall effect." It is the latter form, (18), which gives rise to the commonly expressed notion of a conductivity perpendicular to B which drops as $1/B^2$ for large B, but it is clear from the comparison of (16) and (18) that one must carefully define one's terms before distinguishing conductivity parallel and perpendicular to a magnetic field. In particular, (16) yields $E_u \cdot J_u = RJ_u^2$ which means that the component of E_u in the direction J_u is independent of B and is given by RJ_u. This is not always so for the case of many fluids since, in general, $R_{\parallel} \neq R_{\perp}$.

In the two-fluid case, if the two fluids consist of electrons and relatively massive ions, and the resultant fluid is approximately neutral, we have $|\gamma_-| \gg \gamma_+$, in which case (17) simplifies to

$$(19) \qquad \begin{cases} \xi = -1/q_- \\ R = \alpha/q_-^2 \end{cases} .$$

From $\rho_- v_- + \rho_+ v_+ = 0$ we conclude that $|v_-| \gg |v_+|$ or $J_u = q_- v_-$, from which

$$\xi J_u \times B = -v_- \times B$$

and (16) becomes

$$(20) \qquad E + u_- \times B = R J_u .$$

We might interpret this formula as implying that the electron fluid sees a purely resistive Ohm's law, and the Hall effect enters as in (18) only by referring E to the mass flow velocity which is essentially the same as the ion velocity.

To continue this purely formal analysis of (4), we may decide to keep certain terms on the left-hand side of this equation in addition to the electromagnetic and frictional terms. For example, keeping the stress terms, we get

$$(21) \qquad \rho_r \frac{\partial P^{ij}}{\partial x^j} - \rho \frac{\partial P_r^{ij}}{\partial x^j} + (\rho q_r - \rho_r q) E_u^i = \rho \sum_s \alpha_{rs} (v_r^i - v_s^i) - [(\rho q_r v_r - \rho_r J_u) \times B]^i$$

instead of (10). The solution for v_r and therefore for J_u is now a linear homogeneous expression in the stress gradients as well as in E_u. For the special case of two fluids and scalar stress tensors, the two pressures being of the same order, Ohm's law is easily seen to take the form,

$$(22) \qquad E_u = \frac{1}{\gamma_1 - \gamma_2} \left(\frac{1}{\rho_1} \nabla p_1 - \frac{1}{\rho_2} \nabla p_2 \right) + RJ_u + \xi J_u \times B$$

or

$$(23) \qquad E + u_- \times B = \nabla p_- / q_- + RJ_u$$

in case $|\gamma_-| >> \gamma_+$.

3. Heuristic Considerations.

Let us first consider the case B = 0. The primary mechanism is collisions and we make the basic assumption that the mean-free-path is small compared to distances over which macroscopic changes are observed or, what is roughly equivalent, the mean collision time is short compared to macroscopic times. First, in the two-fluid case, we write (9) in the form

$$(24) \qquad \left\{ \begin{aligned} & \frac{dJ_u}{dt} + Q = - J_u/\tau \\ & \frac{d}{dt} = \frac{\partial}{\partial t} + (u \cdot \nabla) \\ & \frac{1}{\tau} = \frac{a\rho}{\rho_1 \rho_2} = \frac{\rho_1 \rho_2}{\rho} (\gamma_1 - \gamma_2)^2 \, R \end{aligned} \right.$$

All the terms which are not explicitly shown are collected together in Q which contains, among others, terms proportional to E and to pressure gradients; τ is on the order of the collision time. If we interpret (24) as an ordinary differential equation for J_u with Q a given constant, we conclude that within a few collision times, τ, J_u approaches the constant τQ, no matter what value J_u had initially. Moreover, if Q is a given function of time which varies slowly compared to τ, a quasi-equilibrium is reached within a few collision times, after which $J_u(t)$ approximates $\tau Q(t)$. Now, the original magneto-hydrodynamic problem is a set of simultaneous partial differential equations. However, if we know (or assume!) that the solution to a given problem for $\rho(x,t)$, $u(x,t)$, $J(x,t)$, etc., and, in particular, for $Q(x,t)$ is such that, moving with the fluid, Q varies slowly, i.e., $\tau dQ/dt$ is, in general, small compared to Q, then we can conclude, for this solution of the set of partial differential equations,

$$J_u \sim \tau Q \quad . \tag{25}$$

Based mainly on analogy with ordinary fluid dynamics, the tacit assumption is made that if τ is short compared to the external time variation of constraints, τ will also be small when compared to the fluid flow time constants.

The basic step is this justification of (25); further simplifications, such as the dropping of all terms in Q except for E, depend on the analysis of representative magnitudes in specific problems.

In the present case, B = 0, we have $E = \nabla\varphi$. Moreover, it frequently happens that $\varepsilon\varphi$ has the same order of magnitude as kT. It follows that $\nabla p/\gamma\rho$ may have the same order as **E**, so that Ohm's law must include the ∇p terms as well as E (cf. (22)). Of course, in special cases, this may not be necessary. For example, one frequently uses Ohm's law for the sole purpose of computing curl E, in this way eliminating E from the remaining equations. If p is a function of ρ (isentropic flow), then curl $(\nabla\rho/\rho) = 0$, so the pressure terms in Ohm's law give no contribution, even though they may not be small. Again, in a linearized problem, curl $(\nabla p/\rho) = \nabla p \times \nabla(1/\rho)$, which is of second order and can usually be dropped.

The many-fluid case is entirely analogous. The simultaneous system (4) for the relative velocities v_r exhibits an exponential decay with possibly several time constants τ_r. The quasi-equilibrium holds not only for J_u but for each of the v_r. It is crucial that we consider the difference momentum equations, (4), rather than the full set, (3), since the total momentum equation for the fluid velocity u does not contain frictional terms and therefore does not exhibit any decay or quasi-equilibrium property.

We remark that the pressure (i.e., concentration) terms alone, without the electromagnetic terms, give the conventional diffusion relations.

Next we allow B to be present while still keeping τ small. We interpret (4) as a set of ordinary differential equations by making explicit the terms dv_r/dt, $a_{rs}(v_r - v_s)$, and $v_r \times B$, while considering all other terms to be slowly varying (equivalent to the previous Q). The time constants of the system (4), as a set of ordinary differential equations, are now complex, but, since τ is small, exactly the same arguments as given above lead to a quasi-equilibrium which is what was termed "Ohm's law" in the formal analysis.

The above analysis has been a bit hasty in calling E slowly-varying. We recall (cf. equations MH-III(15), (16)) that oscillations of E are coupled with those of J, specifically at the plasma frequency in the direction of B and at the gyro-plasma frequency perpendicular to B. However, these oscillations are intrinsically coupled with the displacement current, $\partial E/\partial t$, in Maxwell's equations. It is generally conceded (this matter will be discussed in detail in a later note) that in a slowly varying problem, displacement current can be ignored. What is probably the case is that, in a given "slowly-varying" problem, the solution is given by superposition of two types of solution. The first is a quasi-equilibrium solution which makes use of Ohm's law and Maxwell's equations without $\partial E/\partial t$. The second consists of linear electromagnetic wave propagation for which the speed of light, plasma frequency, etc., are relevant parameters. This linear problem will, in general, involve differential equations with non-constant coefficients which are determined by the solution of the first problem. However, if we are interested only in the gross features of the flow, we need only solve the first, quasi-equilibrium part. We should keep in mind, however, that it is the "high-frequency" component which we disregard that is ultimately responsible for setting up the quasi-equilibrium.

This situation is somewhat similar to that in ordinary gas dynamics in which we can frequently separate a given flow problem into a slowly varying potential flow with "sound waves" superposed. It is the rapid passage of sound waves to and fro which sets up a quasi-equilibrium.

The decision as to whether pressure terms should be kept in Ohm's law is complicated by the pressure of B. Writing $B = \text{curl } A$ we have $E = \nabla\varphi - \partial A/\partial t$ using Maxwell's equations. Although the separation of E into "electrostatic" and "inductive" parts is not unique, the orders of magnitude are more or less determined by the problem. The previous analysis indicates that the pressure terms are comparable to $\nabla\varphi$. However, if B is large, it may very well be that the pressure terms are negligible when compared to E and u x B, i.e., when inserted into Ohm's law. To be more precise, the condition "large B" is ambiguous. We may, for example, let B become large by superposing a large constant field on the flow-keeping J, p, and u fixed in order of magnitude. The limiting form of Ohm's law is then

$$E + u \times B = 0 \tag{26}$$

since the resistive as well as pressure terms become relatively small. On the other hand, we can increase B proportionately by multiplying it by a large constant, while keeping p constant. In this case J increases as B, and we get the form (16) of Ohm's law. Still another possibility is to increase p as B^2; (roughly speaking this is the proper domain of magneto-hydrodynamics, i.e., with magnetic and internal energies comparable). If this is done while keeping density fixed, the pressure terms become dominant in Ohm's law; if this is done keeping temperature constant, the pressure terms drop out of Ohm's law.

Finally, we turn to the case where B is present and large, with the mean-free-path left open. Specifically, we assume that the Larmor radius λ (cf. MH-III, page 14) is small compared to macroscopic dimensions and the gyro-frequency ω is correspondingly large. We need only consider large mean-free-path compared to λ, since, otherwise, this reduces to previous cases. Accordingly, we ignore friction. This time it becomes necessary to consider the full system (3) rather than (4), and, just as before, we interpret the system (3) as a set of ordinary differential equations. Since we are ignoring friction we find a purely oscillating homogeneous solution. The same set of ordinary differential equations with a slowly varying inhomogeneous term has an asymptotic solution which is a sum of two terms -- that given by formally dropping the time derivative (i.e., Ohm's law with the entire Q inserted) and an oscillating term. If we consider the system to be impulsively started (i.e., Q(t) suddenly jumps as a Heaviside function), then the slowly varying and oscillating components of the solution have the same order of magnitude. However, if we start with a quiescent flow which slowly evolves in time, the magnitude of the oscillating term will remain small, giving Ohm's law plus a ripple. In the previous case, with friction, the approach to a quasi-equilibrium was produced by the friction, with plasma and gyro oscillations present as a complication. In the present case, it is the plasma and gyro oscillations which provide the quasi-equilibrium.

In the present analysis, we use (3) and in the formal derivation of Ohm's law we used (4), but the results are clearly the same for computation of the quasi-equilibrium (in which the total velocity, u, enters as a parameter). We would have to use the full set, (3), if we were to compute the oscillating part of the solution.

We conclude with a few remarks about a perfect conductor which we consider to be defined by (26), $E + u \times B = 0$. First, in the friction dominated case, we require sufficiently large R to set up a quasi-equilibrium. Then we reverse ourselves and assume that R is small! Fortunately, there exists a non-negligible domain of the relevant physical parameters for which R is, at one and the same time, large for one purpose and small for the other. The neglect of the Hall vector is frequently justifiable on the basis of the analysis leading to (20). Under conditions which were discussed in MH-III, page 14, we may frequently take $u \sim u_+ \sim u_-$.

The plasma- and gyro-oscillation dominated case can be analyzed in the same way that led originally to (26); this carries the validity of the perfect conductor equation from the domain of "some friction but not too much" to that of "negligible friction," provided that there is a large enough magnetic field.

Appendix

The derivation of (15) is as follows. For $B >> \alpha$, we employ an expansion in powers of α; to first order (10) is

(A-1) $(\rho q_r - \rho_r q)E_u + (\rho q_r v_r - \rho_r J_u) \text{ x } B = 0$.

Multiplying through by $1/\gamma_r = \rho_r/q_r$ and summing,

$$(\rho^2 - q \sum \frac{\rho_r}{\gamma_r})E_u - (\sum \frac{\rho_r}{\gamma_r}))J_u \text{ x } B = 0 \quad ,$$

from which follows the value of ξ in (15). Returning to (A-1), the first order approximation to v_r is found to be

(A-2) $$v_r = - \frac{J \text{ x } B}{B^2 q_r \xi_r} \quad , \quad \frac{1}{\xi_r} = \frac{\rho_r \rho}{\sum \rho_r/\gamma_r} - q_r \quad .$$

The second term in the expansion of J_u is found by inserting these values in (10), and the value of $R_\perp$ follows.

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